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2007 J. Phys.: Condens. Matter 19 356221

(http://iopscience.iop.org/0953-8984/19/35/356221)

J. Phys.: Condens. Matter **19** (2007) 356221 (16pp) doi:10.1088/0953-8984/19/35/356221

Optical transmission through generalized SML superlattices

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Received 14 April 2007, in final form 23 July 2007
Published 20 August 2007
Online at stacks.iop.org/JPhysCM/19/356221

Abstract
The SML model is a kind of one-dimensional (1D) quasiperiodic three-tile structure proposed by Liu *et al* (1991 *Phys. Rev.* B **43** 13240; 1991 *Chin. Phys. Lett.* **8** 533) based on the experimental results of Al–Si–Mn quasicrystals (Boissieu *et al* 1990 *J. Phys.: Condens. Matter* **2** 2499). Generalized SML models (GSML(m, n)) are extensions of the SML one. In this paper, we study the transmission properties of light through GSML(m, n) aperiodic superlattices and use the transparent-component-decimation (TCD) method to analyze the relationship between the characteristics of structures and transmission coefficients. It is found that the propagation matrices (PMs) and transmission coefficients (TCs) for GSML(m, n) multilayers exhibit interesting properties: (1) when n is even, they are all constant; (2) when n is odd but m is even, the former exhibit a six-cycle property while the latter possess a three-cycle feature; (3) when n is odd and $m = 1$, they both possess a pseudo-seven-cycle characteristic. The analytic results are confirmed by numerical evaluations.

1. Introduction

Since the quasicrystal was discovered in experiments by Shechtman *et al* [1], much attention has been devoted to investigating the structures and properties of quasiperiodic systems. Generally, most of the quasiperiodic models which have been extensively studied are of the two-tile type, e.g. the Fibonacci one [2–4], the Thue–Morse one (TM) [5, 6], the generalized Thue–Morse ones (GTM(m, n)) [7, 8], the Fibonacci-class ones (FC(n)) [9–11], the generalized Fibonacci ones (GF(m, n)) [12] and the Penrose one [13] etc.

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0953-8984/07/356221+16$30.00

However, in 1990 Boissieu *et al* [14] reported their important experimental results on Al–Si–Mn quasicrystals. Their diffraction data confirmed that the $A3_{\mathrm{perp}}$ must have 'parallel' components in physical space, which makes the three-dimensional (3D) structure description in terms of Penrose tiling not really reliable, indicating that quasicrystals should have a three-unit-cell structure. A three-element model is appropriate to describe their electronic, transmission and some other properties. Therefore, as an extension of two-tile Fibonacci quasilattices, a three-tile Fibonacci-like model could represent some reality and could be used in potential theoretical applications. Based on this result Liu *et al* [15, 16] suggested the simplest one-dimensional (1D) quasiperiodic three-tile model, the SML one, and studied its spectral structure. Deng *et al* [17] researched the electronic properties and found that the electronic energy spectra are also Cantor-like but have different self-similarity from that of the Fibonacci lattice. Zhou *et al* [18] investigated the electronic localization and obtained that the lattice structure is quasiperiodic but the energy spectrum is of pure point and the electronic states are all localized, which exhibits the characteristics of disordered systems. Luan [19] explored some invariants of the 1D SML. Recently, Chen *et al* [20] investigated the transmission properties of light through SML multilayers and found that the propagation matrices (PMs) exhibit a pseudo-seven-cycle feature and transmission coefficients (TCs) possess quasi-seven-cycle properties.

It is generally known that the Fibonacci model has been extended to GF(m, n) [12] and FC(n) ones [9–11], respectively, and some new interesting mathematical and physical characteristics have been found. Fibonacci, FC(n) and GF$(m, 1)$ models are all typical quasiperiodic patterns, but the GF$(1, 2)$ sequence is a critical model between the quasiperiodic and the nonquasiperiodic one, and GF$(1, n)$ $(n \geqslant 3)$ chains are nonquasiperiodic patterns. Their optical, electronic, and thermic properties are quite different from each other. It is similar in the case of TM and GTM(m, n) models. Naturally, one would ask about the case of the SML sequence and its extension, GSML(m, n) patterns.

In this paper, we study the optical transmission through GSML(m, n) superlattices vertically and find that the transmission features of these systems are quite different from those of SML multilayers, where the latter is a particular case of the former with $m = n = 1$. From the following sections one can see that when m and n are arbitrary positive integers the latter's quasi-seven-cycle property of TCs will be broken, but meanwhile the tristable and three-cycle characteristics for the former will occur. Generally, when n is even, the TCs for GSML$(m, 2q)$ are all constant; when n is odd and m is even, the TCs for GSML$(2p, 2q - 1)$ possess a three-cycle feature; when n is odd and $m = 1$, the TCs for GSML$(1, 2q - 1)$ possess a pseudo-seven-cycle characteristic formally and will decrease to zero with the increase of the generation rapidly. The tristable, three-cycle and opaque characteristics would be useful for the designing of optical tristate devices, filters, switches, and other optical devices.

We organize this paper as follows: section 2 is devoted to introducing the substitution rules of GSML(m, n) sequences. The basic optical transmission theory has been presented in section 3. In section 4, we obtain analytically the PMs and TCs, use the TCD method to discuss the results, and compare them with numerical evaluations. Finally, a brief summary is given in section 5.

2. GSML(m, n) models

The substitution rules of the GSML(m, n) models act as follows: $S \to M$, $M \to L$, and $L \to L^m S^n$, where m and n are all positive integers. S, M and L correspond to three kinds of atomic spacing: short, medium and long, respectively. Starting with an S, the first four generations of GSML(m, n) are

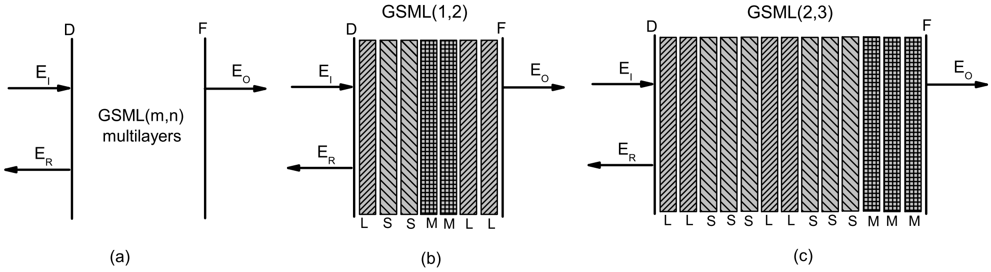

Figure 1. Propagation of light through the reflective and the transmissive surfaces D and F of GSML(m, n) multilayers, where E_I, E_R, and E_O are the input, reflective, and output electromagnetic fields, respectively. (a) The case of general GSML(m, n) systems, and as examples (b) shows that of the sixth generation for GSML(1, 2) systems, and (c) shows the fifth generation for GSML(2, 3) systems.

$$\begin{aligned} G_1 &= S \\ G_2 &= M \\ G_3 &= L \\ G_4 &= L^m S^n, \end{aligned} \tag{1}$$

which show the following recursion relation:

$$G_j = G_{j-1}^m G_{j-3}^n, \qquad (j \geqslant 4). \tag{2}$$

When $m = n = 1$, GSML(m, n) models decay to the SML one.

3. Optical transmission theory

Propagation of light through GSML(m, n) multilayers is illustrated in figure 1, where the system is sandwiched by two media of material of type L, and figures 1(b) and (c) are examples for the sixth generation of GSML(1, 2) and the fifth generation of GSML(2, 3), respectively. By use of the recursion relations (1) and (2), one can obtain the PMs through GSML(m, n) systems as follows:

$$\begin{aligned} M_{G_1} &= M_{LS} M_S M_{SL} \\ M_{G_2} &= M_{LM} M_M M_{ML} \\ M_{G_3} &= M_L \\ M_{G_j} &= M_{G_{j-1}}^m M_{G_{j-3}}^n, \qquad (j \geqslant 4) \end{aligned} \tag{3}$$

where M_{ab} $(a, b = S, M, L)$ stands for the propagation matrix from layer a to layer b, and M_a $(a = S, M, L)$ is that through a single layer a.

For general incidence the TC T_{G_j} can be written as

$$T_{G_j} = \frac{|E_O|^2}{|E_I|^2} = \frac{4(m_{11}m_{22} - m_{12}m_{21})}{(m_{11} + m_{22})^2 + (m_{12} - m_{21})^2}, \tag{4}$$

where E_O and E_I are the output and input electromagnetic fields, respectively, and m_{pq} $(p, q = 1, 2)$ is the element of the matrix M_{G_j}. As M_{G_j} is a second-order unimodular matrix, equation (4) can be rewritten as follows:

$$T_{G_j} = \frac{4}{m_{11}^2 + m_{12}^2 + m_{21}^2 + m_{22}^2 + 2}. \tag{5}$$

On the other hand, we define the ratio of refractive indices as parameter R as follows:

$$R_1 = \frac{n_L}{n_S}, \qquad R_2 = \frac{n_L}{n_M}, \tag{6}$$

where n_S, n_M and n_L are the refractive indices for three media, S, M and L, respectively. Another parameter, the quasi-phase δ, which is the optical phase difference between the ends of a layer [9, 22], is denoted as follows:

$$\delta_i = \frac{n_i k d_i}{\cos\theta_i}, \qquad (i = S, M, L), \tag{7}$$

where k is the wavenumber in vacuum, d_i is the thickness of layer i, and θ_i is the incident angle in layer i. By means of the electromagnetic wave theory one can obtain a set of basic PMs as follows [9, 22]:

$$\begin{aligned}
M_i &= \begin{pmatrix} \cos\delta_i & -\sin\delta_i \\ \sin\delta_i & \cos\delta_i \end{pmatrix}, \qquad (i = S, M, L) \\
M_{SL} = M_{LS}^{-1} &= \begin{pmatrix} 1 & 0 \\ 0 & R_1 \dfrac{\delta_S}{\delta_L} \dfrac{n_L d_L}{n_S d_S} \end{pmatrix} \\
M_{ML} = M_{LM}^{-1} &= \begin{pmatrix} 1 & 0 \\ 0 & R_2 \dfrac{\delta_M}{\delta_L} \dfrac{n_L d_L}{n_M d_M} \end{pmatrix} \\
M_{MS} = M_{SM}^{-1} &= \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{R_2}{R_1} \dfrac{\delta_M}{\delta_S} \dfrac{n_S d_S}{n_M d_M} \end{pmatrix} \\
M_{ii} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I, \qquad (i = S, M, L).
\end{aligned} \tag{8}$$

Usually, people consider the simplest experimental setting and take the incident light to be normal (i.e., $\theta = \theta_i = 0$, $i = S, M, L$). On the other hand, in order to make the quasiperiodicity of the multilayer systems most effective, wavelengths should satisfy the following quasiresonance condition [22]:

$$\delta = \left(l + \tfrac{1}{2}\right)\pi, \tag{9}$$

where l is an integer. In this paper, we choose

$$\begin{aligned} \theta = \theta_i = 0, \\ \delta = \delta_i = \pi/2, \end{aligned} \qquad (i = S, M, L). \tag{10}$$

With this condition, one can rewrite equation (8) as follows:

$$M_i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad (i = S, M, L)$$

$$M_{SL} = M_{LS}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & R_1 \end{pmatrix}$$

$$M_{ML} = M_{LM}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & R_2 \end{pmatrix} \tag{11}$$

$$M_{MS} = M_{SM}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{R_2}{R_1} \end{pmatrix}$$

$$M_{ii} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I, \qquad (i = S, M, L).$$

Additionally, in order to simplify formulae below, we define some mathematical functions as follows:

$$P(x) = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$$

$$Q(x) = \begin{pmatrix} 0 & -x \\ x^{-1} & 0 \end{pmatrix} \tag{12}$$

$$W(x) = \frac{4}{2 + x^2 + x^{-2}}.$$

4. Results and discussion

We present our results of transmission properties for GSML(m, n) systems as the following four cases.

4.1. Even m and even n (i.e. GSML$(2p, 2q)$) systems

When m and n are all even, by use of the aforesaid theory in section 3 one can obtain the PMs as

$$M_{G_j} = \begin{cases} Q(R_1), & (j = 1) \\ Q(R_2), & (j = 2) \\ Q(1), & (j = 3) \\ (-1)^{p+q} I, & (j = 4) \\ (-1)^q I, & (j = 5) \\ (-1)^q I, & (j = 6) \\ I, & (j \geqslant 7), \end{cases} \tag{13}$$

and the corresponding TCs can be expressed as follows:

$$T_{G_j} = \begin{cases} W(R_1) = C_1, & (j = 1) \\ W(R_2) = C_2, & (j = 2) \\ W(1) = 1.0, & (j \geqslant 3). \end{cases} \tag{14}$$

This means that the transmission for GSML$(2p, 2q)$ is tristable, and from the third generation on this kind of system will be transparent.

In order to explain the aforementioned results, we analyze the characteristics of the systems' structures by means of our recently proposed TCD method. From the substitution rules of GSML(m, n) (see equations (1) and (2)) one can see that, if m and n are all even (i.e., $m = 2p$, $n = 2q$) and from the fourth generation on the GSML$(2p, 2q)$ sequences must be composed of the following three kinds of string: $A = S^{2i}$, $B = M^{2j}$, $C = L^{2k}$, where $i, j, k = p, q$. By means of the TCD method [21] one knows that an even layer of medium i (ELMI, i.e., i^{2p}, $i = S, M, L$) is a transparent cell and can be decimated from the multilayers when we calculate the optical transmission effect of the system. Here we show the decimation procedure by the TCD method as follows:

$$\begin{aligned} G_1 &= S \\ G_2 &= M \\ G_3 &= L \\ G_4 &= G_3^{2p} G_1^{2q} \\ &= L^{2p} S^{2q} \\ &\Leftrightarrow \Xi \\ G_5 &= G_4^{2p} G_2^{2q} \\ &\Leftrightarrow \Xi M^{2q} \\ &\Leftrightarrow \Xi \\ G_6 &= G_5^{2p} G_3^{2q} \\ &\Leftrightarrow \Xi\Xi \\ &\Leftrightarrow \Xi \\ &\cdots\cdots, \end{aligned} \tag{15}$$

where '$\Leftrightarrow$' means 'is equivalent to' and Ξ denotes a cell transparent to the substrates of medium i ($i = S, M, L$) and can be decimated completely from the sequence. Obviously, one can see that, when m and n are all even and from the fourth generation on the total systems will be transparent and the TCs are equal to 1.0.

By means of equations (3) and (5)–(8), one can calculate the TCs (T) as a function of the quasi-phase (δ) for the GSML(m, n) systems. The numerical evaluations for the GSML(2, 2) system are illustrated in figure 2, where figures 2(a)–(f) are for the first to sixth generations, respectively. One can see that, where $\delta/\pi = 0.5$ (i.e., $\delta = \pi/2$), the TCs in figures 2(a) and (b) are equal to 0.4756 and 0.8521, respectively, and those for the other figure parts all equal to 1.0, respectively. On the other hand, by use of equations (6), (12), and (14) and when n_S, n_M, and n_L are chosen to be 1.2, 2.0, and 3.0, one can deduce that $R_1 = 2.5$, $R_2 = 1.5$, $C_1 = 0.4756$, and $C_2 = 0.8521$. Obviously, the analytical results are confirmed by the numerical evaluations.

On the other hand, from figure 2 it is also seen that, with the increment of generation, the transmission spectrum tends to be made up of perfect band gaps and transparent peaks. This means that when the number of layers of GSML$(2p, 2q)$ superlattices grows, the transmission with different δ values will be transparent or opaque. In figure 2(f) there are some narrow transparent transmission peaks and wide band gaps. This would have potential applications for designing optical filters, optical switches, and some other optical devices. Similar cases appear in other GSML (m, n) systems (see figures 3–6).

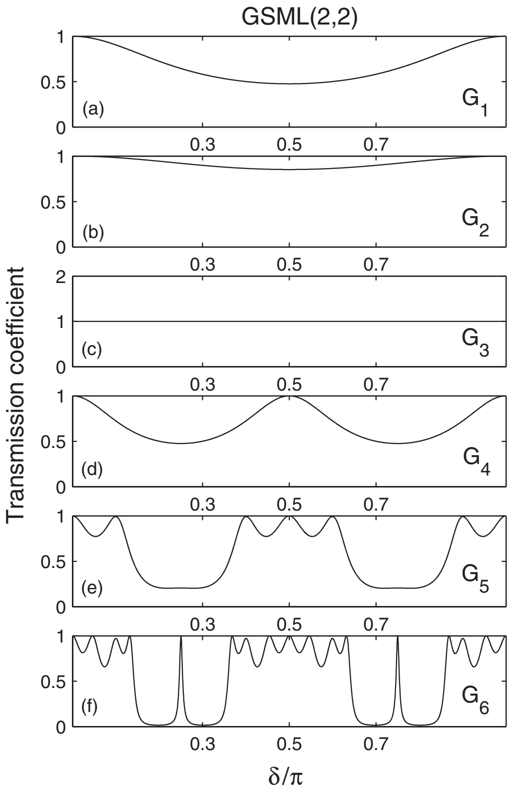

Figure 2. The relationship between transmission coefficient T and quasi-phase difference δ/π for the GSML(2, 2) system, where $n_S = 1.2$, $n_M = 2.0$, $n_L = 3.0$, and the generation is from 1 to 6.

4.2. Odd m and even n (i.e. GSML($2p-1, 2q$)) systems

When m is odd and n is even, similarly to section 4.1, one can deduce that the PMs for GSML($2p-1, 2q$) systems can be written as

$$M_{G_j} = \begin{cases} Q(R_1), & (j=1) \\ Q(R_2), & (j=2) \\ Q(1), & (j=3) \\ (-1)^{p+q}Q(1), & (j=2i, i \geqslant 2) \\ Q(1), & (j=2i+1, i \geqslant 2), \end{cases} \tag{16}$$

and the corresponding TCs can be expressed in the same way as in equation (14); that is to say, if medium L is used as the substrates then the transmission properties for the cases of sections 4.1 and 4.2 are all the same.

Here we also use the TCD method to discuss the optical transmission properties of the sequences qualitatively. The decimation procedure by the TCD method can be expressed as

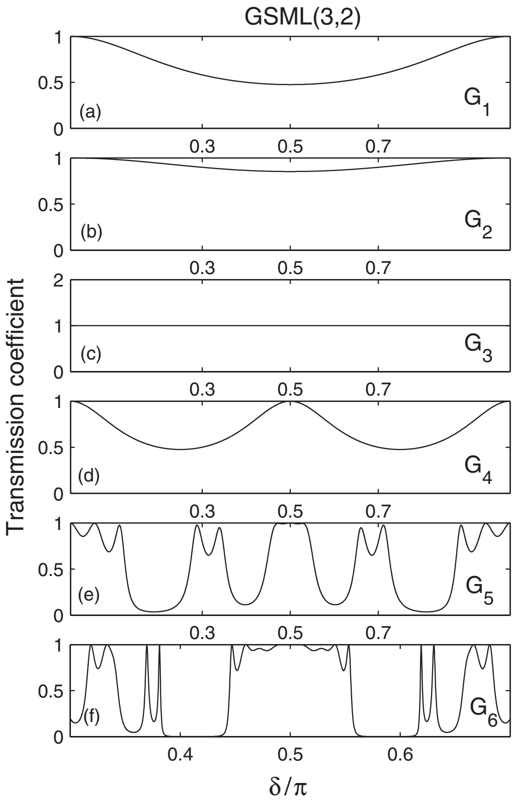

Figure 3. The relationship between transmission coefficient T and quasi-phase difference δ/π for the GSML(3, 2) system, where $n_S = 1.2$, $n_M = 2.0$, $n_L = 3.0$, and the generation is from 1 to 6.

follows:

$$\begin{aligned} G_1 &= S \\ G_2 &= M \\ G_3 &= L \\ G_4 &= G_3^{2p-1} G_1^{2q} \\ &= L^{2p-1} S^{2q} \\ &\Leftrightarrow \Xi \\ G_5 &= G_4^{2p-1} G_2^{2q} \\ &\Leftrightarrow \Xi M^{2q} \\ &\Leftrightarrow \Xi \\ G_6 &= G_5^{2p-1} G_3^{2q} \\ &\Leftrightarrow \Xi\Xi \\ &\Leftrightarrow \Xi \\ &\cdots\cdots, \end{aligned} \tag{17}$$

where the component L^{2p-1} is transparent to the substrates of medium L. Consequently, the qualitative conclusion is accordant with the analytical formulae.

The numerical results for the first six generations of the GSML(3, 2) system are shown in figure 3. One can see that when $\delta/\pi = 0.5$ (i.e. $\delta = \pi/2$) the TCs in figures 3(a) and (b)

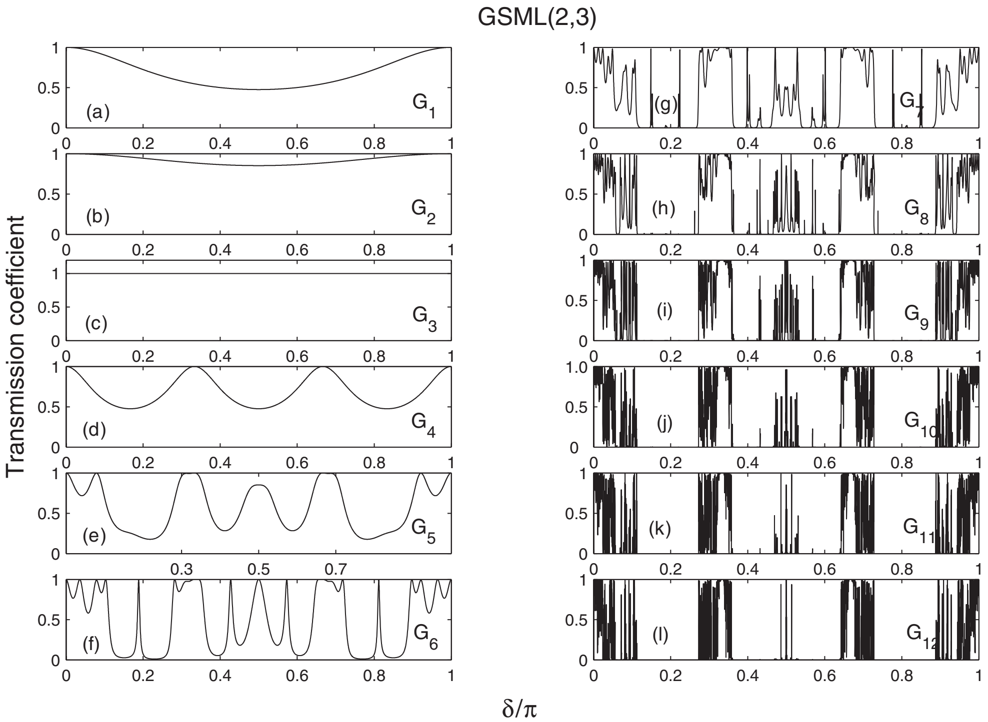

Figure 4. The relationship between transmission coefficient T and quasi-phase difference δ/π for the GSML(2, 3) system, where $n_S = 1.2$, $n_M = 2.0$, $n_L = 3.0$, and the generation is from 1 to 12.

are equal to 0.4756 and 0.8521, respectively, and those for the other figure parts are all equal to 1.0, respectively. The cases are the same as those of the GSML(2, 2) system although the total curves of figures 3(e) and (f) are somewhat different from those of figures 2(e) and (f). Obviously, the analytical results are confirmed by the numerical evaluations.

4.3. Even m and odd n (i.e. GSML($2p, 2q-1$)) systems

With this condition, it is found that the PMs are six-cycle, as follows:

$$M_{G_j} = \begin{cases} Q(R_1), & (j = 6i+1) \\ Q(R_2), & (j = 6i+2) \\ Q(1), & (j = 6i+3) \\ (-1)^{p+q} Q(R_1), & (j = 6i+4) \\ (-1)^{p+q} Q(R_2), & (j = 6i+5) \\ (-1)^{p+q} Q(1), & (j = 6i+6), \end{cases} \tag{18}$$

where $i \geqslant 0$. Similarly, one can also obtain the corresponding TCs as follows:

$$T_{G_j} = \begin{cases} C_1, & (j = 3i+1) \\ C_2, & (j = 3i+2) \\ 1.0, & (j = 3i+3), \end{cases} \tag{19}$$

where $i \geqslant 0$. Obviously, the TCs exhibit a three-cycle feature. The original reason is the symmetry of the aperiodic sequence.

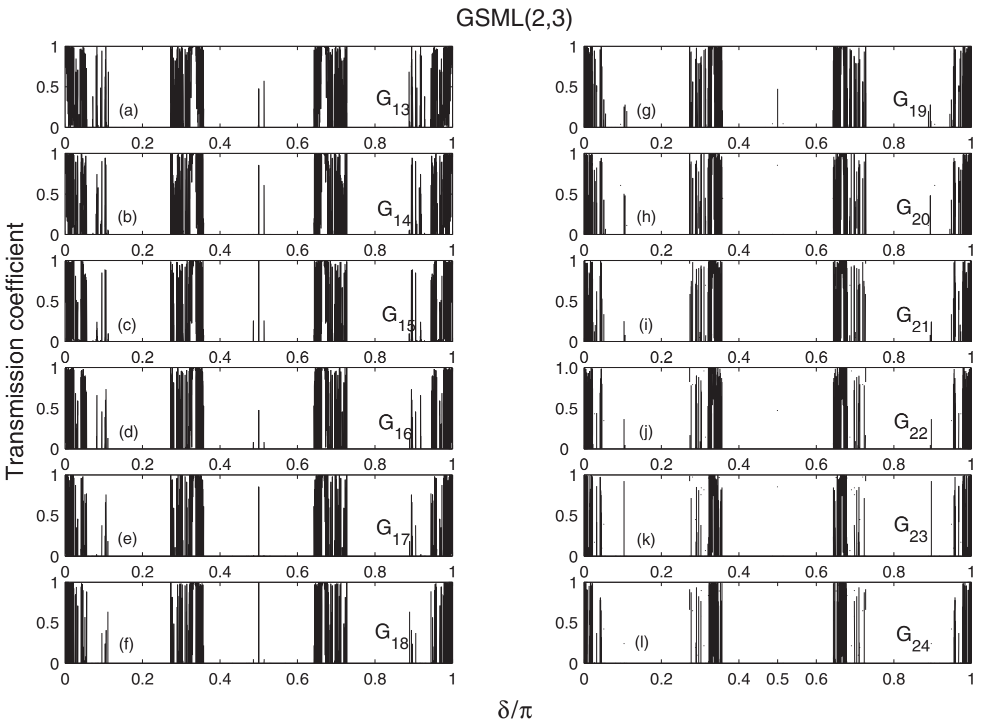

Figure 5. The relationship between transmission coefficient T and quasi-phase difference δ/π for the GSML(2, 3) system, where $n_S = 1.2$, $n_M = 2.0$, $n_L = 3.0$, and the generation is from 13 to 24.

Also using the TCD method, from the fourth generation onwards, one can simplify the GSML($2p, 2q-1$) sequence again and again until it has the simplest form. The decimation procedure can be expressed as follows:

$$\begin{aligned}
G_4 &= G_3^{2p} G_1^{2q-1} \\
&= L^{2p} S^{2q-1} \\
&\Leftrightarrow S \\
&= G_1 \\
G_5 &= G_4^{2p} G_2^{2q-1} \\
&\Leftrightarrow S^{2p} M^{2q-1} \\
&\Leftrightarrow M \\
&= G_2 \\
G_6 &= G_5^{2p} G_3^{2q-1} \\
&\Leftrightarrow M^{2p} L^{2q-1} \\
&\Leftrightarrow L \\
&= G_3 \\
&\cdots\cdots,
\end{aligned} \tag{20}$$

Obviously, one can deduce that

$$\begin{aligned}
G_{3i+1} &\Leftrightarrow S \Rightarrow T_{3i+1} = T_{L-S-L} = C_1 \\
G_{3i+2} &\Leftrightarrow M \Rightarrow T_{3i+2} = T_{L-M-L} = C_2 \\
G_{3i+3} &\Leftrightarrow L \Rightarrow T_{3i+3} = T_{L-L-L} = 1.0,
\end{aligned} \tag{21}$$

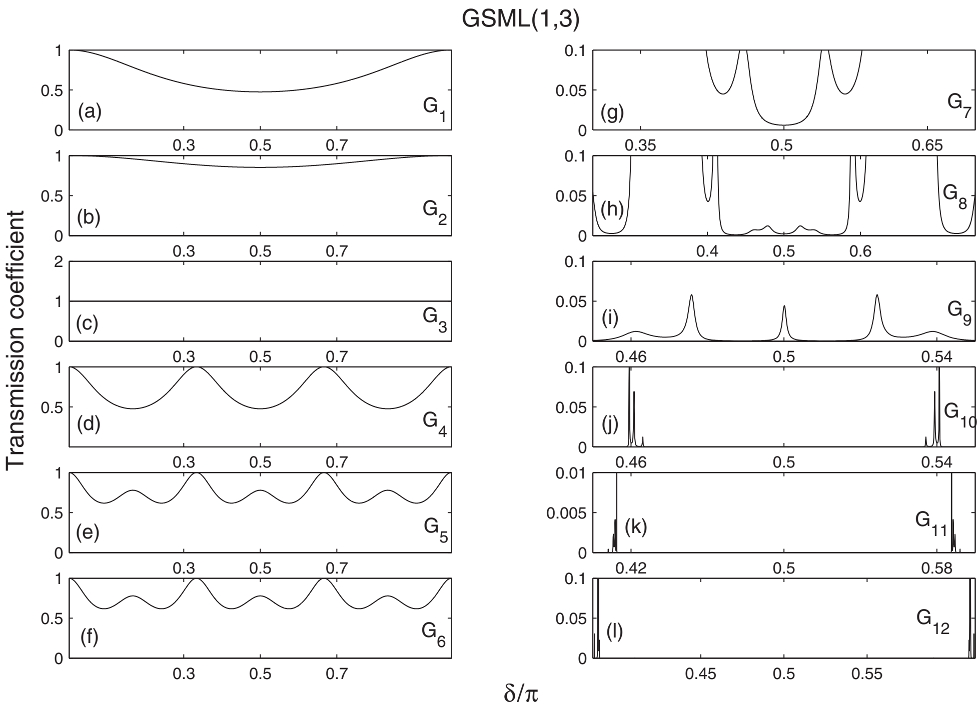

Figure 6. The relationship between transmission coefficient T and quasi-phase difference δ/π for the GSML(1, 3) system, where $n_S = 1.2$, $n_M = 2.0$, $n_L = 3.0$, and the generation is from 1 to 12.

where $i \geqslant 0$. From equation (21) one can find that the optical GSML($2p, 2q - 1$) aperiodic superlattices are composed of some ELMI cells, which are all transparent components, and some other nontransparent ones. After all of the ELMI cells have been decimated, the sequence will be cyclic with the increase of generation. In brief, the cyclic property of the optical transmission for GSML($2p, 2q - 1$) multilayers is caused by the sequence symmetry.

The numerical results of the first 24 generations for the GSML(2, 3) system are shown in figures 4 and 5. One can see that when $\delta/\pi = 0.5$ the TCs in figures 4(a), (d), (g), and (j) and 5(a), (d), (g), and (j) are all equal to 0.4756; those in figures 4(b), (e), (h), and (k) and 5(b), (e), (h), and (k) are all equal to 0.8521; and those in figures 4(c), (f), (i), and (l) and 5(c), (f), (i), and (l) are all equal to 1.0. This shows that the transmission for the GSML(2, 3) system possesses a three-cycle *low–middle–high* tristable property. On the other hand, it can be seen that in figures 4 and 5 there are some very sharp bright peaks around $\delta/\pi = 0.5$, and the zone tends to be narrower and narrower with the increase of generations. In figures 5(h)–(l), there exists only one nonzero point in the zone of $\delta/\pi \approx 0.356$–0.644. This means that in this zone there exists only one delocalized mode over the whole sample, which is at the point of $\delta/\pi = 0.5$, and the others are all localized ones. The reason is that at the central point the wavelength satisfies the resonance condition and the optical phase is matched, so the transmission is stable and there is no further attenuation even if the number of system layers increases rapidly with the growth of the generation. In contrast, at other points in the zone, the resonance condition is broken up, the attenuation effect accumulates seriously after light transmits through many layers and the modes are all localized. Additionally, from equation (7) one knows that $\delta \sim k \sim 1/\lambda$, then taking advantage of this transmissive characteristic we could produce a perfect optical filter for sorting out some light with specific wavelengths. This would be useful for designing some optical devices.

4.4. $m = 1$ and odd n (i.e., GSML$(1, 2q-1)$) systems

When m and n are all odd, unfortunately, we have not obtained the general formulae, but only have some special results for the case of $m = 1$.

When $m = n = 1$, GSML(m, n) models decay to the SML one, for which the transmission properties have been studied in detail by Chen *et al* [20].

When $m = 1$, and n is odd (i.e., GSML$(1, 2q-1)$), one can obtain the results for the first five generations as follows:

$$M_{G_j} = \begin{cases} Q(R_1), & (j=1) \\ Q(R_2), & (j=2) \\ Q(1), & (j=3) \\ (-1)^{q-1} P(-R_1^{-1}), & (j=4) \\ Q(-R_1^{-1} R_2), & (j=5), \end{cases} \tag{22}$$

$$T_{G_j} = \begin{cases} C_1, & (j=1) \\ C_2, & (j=2) \\ 1.0, & (j=3) \\ C_1, & (j=4) \\ W(R_1 R_2^{-1}) = C_3, & (j=5). \end{cases} \tag{23}$$

However, from the sixth generation on, the PMs for GSML$(1, 2q-1)$ systems will have a pseudo-seven-cycle feature without taking into account the signs, while their signs show a 14-cycle characteristic. The corresponding TCs possess a quasi-seven-cycle property. The signs of the matrices can be expressed as follows:

$$\mathrm{Sign}(M_{G_j}) = \begin{cases} (-1)^{q-1}, & (j = 14i+6, i \geqslant 0) \\ (-1)^{q-1}, & (j = 14i+8, i \geqslant 0) \\ (-1)^{q-1}, & (j = 14i+12, i \geqslant 0) \\ (-1)^{q-1}, & (j = 14i+4, i \geqslant 1) \\ +1, & \text{(others)}. \end{cases} \tag{24}$$

The PMs without taking into account the signs can be written as follows:

$$M_{G_j} = \begin{cases} P(R_1^{x_{7j-1}} R_2^{y_{7j-1}}), & (j = 7i-1) \\ P(R_1^{x_{7j}} R_2^{y_{7j}}), & (j = 7i) \\ Q(R_1^{x_{7j+1}} R_2^{y_{7j+1}}), & (j = 7i+1) \\ Q(R_1^{x_{7j+2}} R_2^{y_{7j+2}}), & (j = 7i+2) \\ Q(R_1^{x_{7j+3}} R_2^{y_{7j+3}}), & (j = 7i+3) \\ P(R_1^{x_{7j+4}} R_2^{y_{7j+4}}), & (j = 7i+4) \\ Q(R_1^{x_{7j+5}} R_2^{y_{7j+5}}), & (j = 7i+5), \end{cases} \tag{25}$$

where $i \geqslant 1$. Then one can obtain the TCs as follows:

$$T_{G_j} = \begin{cases} W(R_1^{x_{7j-1}} R_2^{y_{7j-1}}), & (j = 7i - 1) \\ W(R_1^{x_{7j}} R_2^{y_{7j}}), & (j = 7i) \\ W(R_1^{x_{7j+1}} R_2^{y_{7j+1}}), & (j = 7i + 1) \\ W(R_1^{x_{7j+2}} R_2^{y_{7j+2}}), & (j = 7i + 2) \\ W(R_1^{x_{7j+3}} R_2^{y_{7j+3}}), & (j = 7i + 3) \\ W(R_1^{x_{7j+4}} R_2^{y_{7j+4}}), & (j = 7i + 4) \\ W(R_1^{x_{7j+5}} R_2^{y_{7j+5}}), & (j = 7i + 5), \end{cases} \tag{26}$$

where $i \geqslant 1$. For the power exponents (x_j, y_j), three groups of them can be directly expressed as

$$\begin{aligned} x_{7j-1} &= (-1)^j n^{j-1} \\ y_{7j-1} &= (-1)^{j+1} n^{j-1}, \end{aligned} \tag{27}$$

$$\begin{aligned} x_{7j} &= (-1)^j \left(n^{2j-1} + 2 \sum_{i=2j-2}^{j} n^i + n^{j-1} \right) \\ y_{7j} &= (-1)^{j+1} \left(2 \sum_{i=2j-2}^{j} n^i + n^{j-1} \right), \end{aligned} \tag{28}$$

$$\begin{aligned} x_{7j+4} &= (-1)^{j+1} \left(n^{2j} + 2 \sum_{i=2j-1}^{j} n^i \right) \\ y_{7j+4} &= (-1)^j \left(2 \sum_{i=2j-1}^{j} n^i \right), \end{aligned} \tag{29}$$

where $j \geqslant 1$, and the other four groups can be indirectly expressed as follows:

$$\begin{aligned} x_{7j+2} &= x_{7j+1} - n x_{7j-1} \\ y_{7j+2} &= y_{7j+1} - n y_{7j-1}, \end{aligned} \qquad (j \geqslant 1), \tag{30}$$

$$\begin{aligned} x_{7j+3} &= x_{7j+2} - n x_{7j} \\ y_{7j+3} &= y_{7j+2} - n y_{7j}, \end{aligned} \qquad (j \geqslant 1), \tag{31}$$

$$\begin{aligned} x_{7j+5} &= x_{7j+4} + x_{7j+2} \\ y_{7j+5} &= y_{7j+4} + y_{7j+2}, \end{aligned} \qquad (j \geqslant 1), \tag{32}$$

and

$$\begin{aligned} x_{7j+1} &= x_{7j} + x_{7j-3} + x_{7j-6} - n x_{7j-8} \\ y_{7j+1} &= y_{7j} + y_{7j-3} + y_{7j-6} - n y_{7j-8}, \end{aligned} \tag{33}$$

where $j \geqslant 2$. In particular, if $j = 1$,

$$\begin{aligned} x_{7j+1} &= x_8 = n + 2 \\ y_{7j+1} &= y_8 = -2. \end{aligned} \tag{34}$$

For example, we calculate the values of (x_j, y_j) $(j = 6, \ldots, 12)$ as follows:

$$\begin{aligned}
(x_6, y_6) &= (-1, 1)\\
(x_7, y_7) &= (-n-1, 1)\\
(x_8, y_8) &= (n+2, -2)\\
(x_9, y_9) &= (2, n-2)\\
(x_{10}, y_{10}) &= (-n^2 - n + 2, 2n - 2)\\
(x_{11}, y_{11}) &= (n^2 + 2n, -2n)\\
(x_{12}, y_{12}) &= (-n^2 - 2n + 2, 3n - 2).
\end{aligned} \tag{35}$$

By means of the aforementioned results and when $n_S = 1.2$, $n_M = 2.0$, and $n_L = 3.0$, one can obtain the TCs for the sixth to the 12th generations as follows:

$$\begin{aligned}
T_{G_6} &= C_3 = 0.7785\\
T_{G_7} &= 0.0059\\
T_{G_8} &= 0.0021\\
T_{G_9} &= 0.0434\\
T_{G_{10}} &\approx 5.2 \times 10^{-9}\\
T_{G_{11}} &\approx 3.4 \times 10^{-12}\\
T_{G_{12}} &\approx 1.6 \times 10^{-12}.
\end{aligned} \tag{36}$$

Figure 6 shows the numerical stimulations for the first 12 generations of GSML(1, 3) systems. One can see that when $\delta/\pi = 0.5 T_1 = 0.4756$, $T_2 = 0.8521$, $T_3 = 1.0$, $T_4 = 0.4756$, $T_5 = 0.7785$, $T_6 = 0.7785$, $T_7 = 0.0059$, $T_8 = 0.0021$, $T_9 = 0.0434$, $T_{10} \approx 0.0$, $T_{11} \approx 0.0$, and $T_{12} \approx 0.0$. On the other hand, from section 4.1 one can also obtain the values of C_1 and C_2. Obviously, the analytical results are confirmed by the numerical evaluations.

In order to summarize the results of optical transmission normally through GSML(m, n) multilayers, we plot the numerical evaluations in figure 7, where $R_1 = 3.0/1.2 = 2.5$, $R_2 = 3.0/2.0 = 1.5$, $C_1 = 0.4756$, and $C_2 = 0.8521$, respectively. From figures 7(a) and (b) one can see that when n is even (i.e. GSML$(2p, 2q)$ and GSML$(2p-1, 2q)$ systems) the TCs for the first and second generations are equal to constants 0.4756 and 0.8521, respectively; from the third generation onwards, the TCs keep constant at 1.0. These are two kinds of tristable optical system. Figure 7(c) shows the case of GSML$(2p, 2q-1)$ systems. The TCs for all of the $3i+1$th generations are equal to constant 0.4756, and those for the $3i+2$th and $3i+3$th generations equal constants 0.8521 and 1.0, respectively. Obviously, this is a kind of three-cycle tristable optical system. Unfortunately, the TCs for GSML$(1, 2q-1)$ systems are very complicated and their values will be quite different from each other, although they can be expressed in a pseudo-seven-cycle formula formally (see equation (26)). As an example, we illustrate the results of the GSML(1, 3) system in figure 7(d). The TCs for the first six generations are equal to 0.4756, 0.8521, 1.0, 0.4756, 0.7785, and 0.7785, respectively. From the seventh generation onwards, the TCs will decrease to zero rapidly (see equation (36)).

5. Summary

We have studied the optical transmission through GSML(m, n) multilayers, for which the substitution rules are $S \to M$, $M \to L$, and $L \to L^m S^n$. The formulae for PMs and TCs have been obtained analytically.

It is found that when n is even (i.e. GSML$(m, 2q)$), for the first and second generation systems, the latter are equal to C_1 and C_2, respectively, but for the other generations the latter

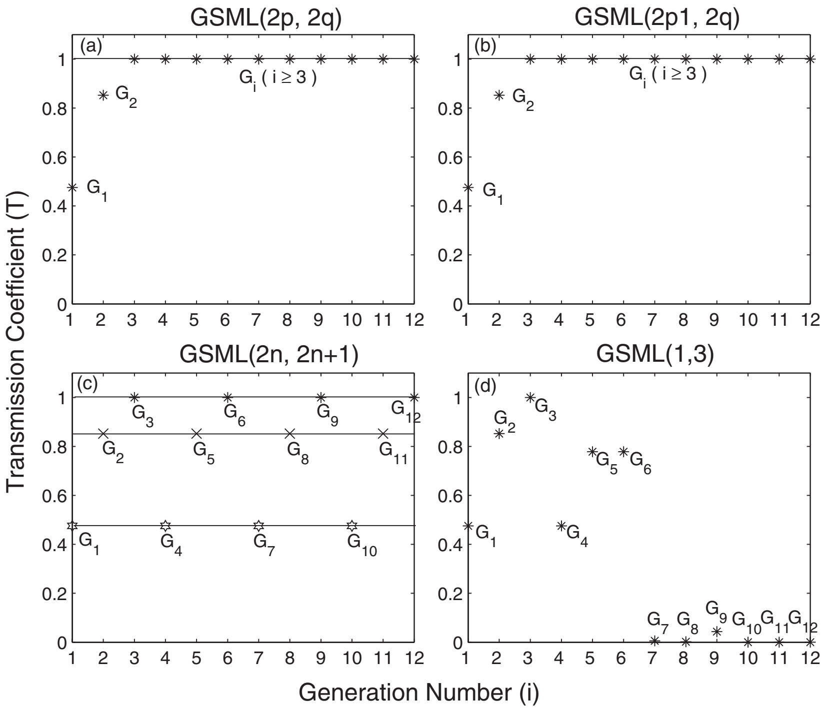

Figure 7. The relationship between transmission coefficient T and the generation number i when light normally transmits through GSML(m, n) multilayers, where $n_S = 1.2$, $n_M = 2.0$, $n_L = 3.0$.

will be equal to a constant 1.0, i.e. the systems are then transparent. When m is even and n is odd (i.e. GSML$(2p, 2q-1)$), the latter display a three-cycle tristable feature (i.e. C_1–C_2–1.0–C_1–C_2–1.0–$\cdots$). When $m = 1$ and n is odd (i.e. GSML$(1, 2q-1)$) and from the sixth generation on, the former possess a pseudo-seven-cycle property, while the latter will decrease to zero rapidly.

The analytical results are confirmed by numerical evaluations. Additionally, we use the TCD method to discuss the results and analyze the structure of the systems. Just because of the symmetry of the structure, the transmission coefficients display cyclic and tristable properties.

Acknowledgment

This work was supported by the National Natural Science Foundation of China, grant No 10004003 and the Program for Innovative Research Team of the Higher Education in Guangdong.

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